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2001 J. Phys. A: Math. Gen. 34 3309

(http://iopscience.iop.org/0305-4470/34/15/309)

INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND GENERAL

J. Phys. A: Math. Gen. **34** (2001) 3309–3324 www.iop.org/Journals/ja PII: S0305-4470(01)17934-5

A geometric basis for the standard-model gauge group

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Received 5 October 2000, in final form 15 January 2001

Abstract

A geometric approach to the standard model in terms of the Clifford algebra $C\ell_7$ is advanced. A key feature of the model is its use of an algebraic spinor for one generation of leptons and quarks. Spinor transformations separate into left-sided ('exterior') and right-sided ('interior') types. By definition, Poincaré transformations are exterior ones. We consider all rotations in the seven-dimensional space that (1) conserve the spacetime components of the particle and antiparticle currents and (2) do not couple the right-chiral neutrino. These rotations comprise additional exterior transformations that commute with the Poincaré group and form the group $SU(2)_L$, interior ones that constitute $SU(3)_C$ and a unique group of coupled double-sided rotations with $U(1)_Y$ symmetry. The spinor mediates a physical coupling of Poincaré and isotopic symmetries within the restrictions of the Coleman–Mandula theorem. The four extra spacelike dimensions in the model form a basis for the Higgs isodoublet field, whose symmetry requires the chirality of $SU(2)$. The charge assignments of both the fundamental fermions and the Higgs boson are produced exactly.

PACS numbers: 1210D, 1110K, 1260, 1140

1. Introduction

This paper introduces a geometric approach to the minimal standard model in terms of Clifford's geometric algebra $C\ell_7$ of seven-dimensional space (see e.g. [1–4] for an introduction to Clifford algebras and their applications in physics). It demonstrates how the gauge symmetries $U(1)_Y \otimes SU(2)_L \otimes SU(3)_C$ arise as the rotational symmetries of a reducible representation of the Poincaré group in a linear space with only four extra spacelike dimensions. The fact that this is fewer than the minimum of seven extra dimensions required in the Kaluza–Klein type of approach [5] stems both from the availability of double-sided transformations on algebraic spinor elements and from the existence of higher-dimensional multivector subspaces in $C\ell_7$. Our approach of studying rotational symmetries in a higher-dimensional space may be viewed as an extension of the well known association of spin with spatial rotations and the treatment of charge symmetry as a rotational symmetry in isospin space. It may lead to a better understanding of the geometry underlying the standard model.

There have been numerous attempts in the past to combine the existing symmetries into an encompassing structure. Many of the earlier ones have fallen victim to theorems such as the one by Coleman and Mandula [6] that disallow most except 'trivial' (i.e. direct-product) couplings of internal and spacetime symmetries of the S matrix. One of the motivations of supersymmetric models has been to evade the restrictions of such theorems [7–9].

More recently Clifford algebras have been used to model the leptons and quarks and their interactions [10–14]. The Clifford algebras $C\ell_3$ of physical space and $C\ell_{1,3}$ of Minkowski spacetime are just the algebra of Pauli spin matrices and the real algebra of Dirac gamma matrices, respectively, and are essential ingredients of the relativistic quantum theory of fermions. The aim is to find a larger algebra containing $C\ell_3$ and $C\ell_{1,3}$ as subalgebras that models several fermions simultaneously. Chisholm and Farwell [13] investigated the mathematical constraints on models in which the particle spinors belong to minimal left ideals of the algebra, and they developed a spin-gauge theory [10] in which couplings to a 'frame field' are responsible for the boson masses. They studied $C\ell_{1,6}$ [11], represented by 8×8 matrices, to model the electroweak interactions of leptons in a seven-dimensional space, $C\ell_{2,6}$ [10], represented by 16×16 matrices, to combine the electroweak and gravitational interactions in an eight-dimensional space, and both $C\ell_{4,7}$ [11] and $C\ell_{3,8}$ [12], represented by 32×32 matrices, to model one generation of eight fermions (without separate antiparticles) in an 11-dimensional space. In all these models, gauge transformations acted on the spinors only from the left. In 1999 they chose a different behaviour [14] for the gauge transformations, in which the spinors, still taken to be minimal left ideals, undergo similarity transformations. They interpreted a new interaction term resulting from this formulation in $C\ell_{1,6}$ as representing the $U(1)$ contribution in electroweak theory.

Our algebraic approach builds on a previous formulation [15] in geometric algebra of the Dirac theory. While it shares many of the powerful tools and algebraic structures of Clifford algebras with the work of Chisholm and Farwell, it is distinct in several respects. Our spinor, representing all the fermions and their antiparticles for a single generation of the standard model, is not an element of a minimal left ideal. However, isotopic pairs of particles can be isolated in the spinor by applying primitive idempotents on the right, and such projected spinors do belong to distinct minimal left ideals. The transformation behaviour is determined by the geometric role of the spinor [1, 15, 16] as an amplitude of the Lorentz transformation relating a reference frame for the particles to the laboratory frame: the spinor is subject to *independent* transformations on the right and left. It is through this structure, together with the Minkowskian metric of paravector space [4], that we are able to model all the fermions of a generation in just seven spatial (eight spacetime) dimensions with an algebra represented by 8×8 matrices. More important than the compactness of our model, however, are its results. Our paper emphasizes the geometrical significance of the algebra. The spinors physically couple 'interior' and 'exterior' symmetries (to be defined below) in such a way as to maintain a direct-product group structure in their two-sided transformations. The $SU(2)$ and $SU(3)$ symmetries arise as the exact exterior and interior rotation groups, respectively, in the seven-dimensional space that (1) conserve the physical spacetime components of particle and antiparticle currents and (2) leave the right-chiral neutrino sterile. The $U(1)$ symmetry is given by coupled rotations that act simultaneously on both sides of the spinor, commute with the interior and exterior rotations and satisfy the constraints (1) and (2). It is important to emphasize that in our geometric model the gauge symmetries are not imposed but arise naturally from the algebra itself as unique symmetry groups of the current. The chiral nature of the $SU(2)$ group is discussed in terms of the symmetry of the Higgs field. The model also predicts the correct weak-hypercharge assignments.

Section 2 summarizes the conventions adopted and provides an 8×8-matrix representation of $C\ell_7$ in order to relate our algebraic formulation to conventional expressions. Section 3 develops the notion of spinors in $C\ell_7$. It shows how spinors representing all the fermions of a single generation can be combined into a single algebraic spinor, how the currents are calculated from such spinors and how the contributions from individual fermions can be projected out. In section 4, we study the rotational symmetries of these spinors and show that they give exactly the gauge symmetries of the standard model with the correct weak hypercharge assignments. We also investigate other possible symmetry transformations and show that within our model the continuous interior and exterior symmetry groups (other than the Poincaré group) comprise only sets of coupled rotations. Section 5 shows how the four extra spatial dimensions and their transformation properties are precisely what is needed for the four components of a minimal Higgs field.

2. Algebraic foundations

Clifford algebras are associative algebras of vectors. In the real Clifford algebra $C\ell_7$, the unit vectors $e_1, e_2, \ldots, e_7$ are chosen to represent orthogonal spacelike directions in the tangent space of a seven-dimensional manifold, with e_1, e_2, e_3 allotted to the three observed (physical) directions. The product of any number of vectors is completely determined by the anticommutator

$$e_j e_k + e_k e_j = 2\delta_{jk} \qquad j, k = 1, \ldots, 7. \tag{1}$$

All elements of the algebra can be reduced to real linear combinations of $2^7 = 128$ basis forms, each one representing a geometric object. For example, the bivector e_1e_4 represents the plane spanned by the directions e_1 and e_4, and the trivector $e_1e_2e_3$ represents the physical volume element. There are a total of 21 independent bivectors and in general $\binom{7}{k}$ independent k-vectors (forms built from products of k distinct basis vectors) in $C\ell_7$.

Two basic conjugations, both of which are antiautomorphic involutions, are used. The reversion of $K \in C\ell_7$, denoted $K^\dagger$, reverses the order of appearance of all vector elements within K. For example, $(e_1e_2e_3)^\dagger = e_3e_2e_1 = -e_1e_2e_3$ and $(AB)^\dagger = B^\dagger A^\dagger$. Clifford conjugation, denoted by $\bar{K}$, both reverses the order and negates all vector elements of K. In the algebras $C\ell_n$, the basis vectors e_j can all be taken to be Hermitian, and then reversion is equivalent to Hermitian conjugation. The algebra $C\ell_7$ is appealing in that the volume element of the algebra, like that of $C\ell_3$, commutes with all elements and squares to -1. It can therefore be associated identically with the imaginary unit

$$\mathrm{i} \equiv e_1e_2e_3e_4e_5e_6e_7 \tag{2}$$

and used to reduce products of real vectors to elements of a complex space with 64 basis forms. For example, $e_4e_5e_6e_7 = -\mathrm{i}e_1e_2e_3$. This fortuitous circumstance occurs for every $C\ell_{3+4n}$ with non-negative integer n, and $C\ell_7$ is the smallest of the series that contains the Dirac algebra as a subalgebra. The choice of adding exactly four extra dimensions to physical space is further justified below in that they arise naturally from a metric-free approach to physical space and form a natural basis for the four components of the minimal Higgs field.

The formalism used here builds on the physical applications of $C\ell_3$ (the Pauli algebra), in particular the use of *paravectors* [4, 17] to model spacetime vectors. Paravectors are sums of scalars and vectors such as $V = V^0 + V^1e_1 + V^2e_2 + V^3e_3 \equiv V^\mu e_\mu$, where for notational convenience we denote the unit scalar by e_0, and the scalar V^0 is the time component in the observer frame, that is, the frame with proper velocity $e_0 = \bar{e}_0 = 1$. The linear space of paravectors has a Minkowski spacetime metric $\eta_{\mu\nu}$ with signature $(1, 3)$. The metric arises from the square norm of paravectors

$$V\bar{V} = \langle V\bar{V}\rangle_{\mathrm{S}} = V^\mu V^\nu \langle e_\mu \bar{e}_\nu\rangle_{\mathrm{S}} = V^\mu V_\mu \tag{3}$$

as $\eta_{\mu\nu} = \langle e_\mu \bar{e}_\nu \rangle_S$. Here, $\langle \cdots \rangle_S$ means the scalar part of the enclosed expression, and we adopt the summation convention for repeated indices, with lower-case Greek indices taking integer values $0 \ldots 3$. The algebra generated by products of paravectors is just $C\ell_3$, which is isomorphic to quaternions over the complex numbers. It admits a covariant formulation of relativity and has also been shown to provide a natural formulation of the single-particle Dirac theory [15]. The Lorentz-invariant spacetime volume element in $C\ell_3$ can be taken to be $e_0 e_1 e_2 e_3 = \pm \mathrm{i}$. The sign indicates the handedness of the spatial basis vectors $\{e_1, e_2, e_3\}$. As we discuss in more detail in the following section, when extra dimensions are present, it is possible to rotate a right-handed spacetime basis into a left-handed one.

Proper and orthochronous Lorentz transformations of spacetime vectors are effected by bilinear transformations of the form [18]

$$V \rightarrow LVL^\dagger \tag{4}$$

where L is any unimodular element: $L\bar{L} = 1$. Every such L can be expressed as the product $L = \exp(w/2)\exp(\theta/2)$ of a spatial rotation $L_R = \exp(\theta/2)$ in the plane of the bivector $\theta = \frac{1}{2}\theta^{jk} e_j \bar{e}_k$ and a pure boost $L_B = \exp(w/2)$ in the direction of the rapidity $w = w^j e_j$ (or, equivalently, as a hyperbolic rotation in the spacetime plane of $w^j e_j \bar{e}_0$). The scalar coefficients satisfy $\theta^{jk} = -\theta^{kj}$ and $w^j = 0 = \theta^{kj}$ for $j > 3$. An advantage of the formalism is that the generators of the transformations have direct physical significance. For example, the generator $e_1\bar{e}_2$ induces a rotation in the $e_1 e_2$ plane. Note that a scalar is not necessarily the time component of some spacetime vector. The mass m of a particle, for example, may be the time component of the momentum p (in units with $c = 1$) in the rest frame, or it may be the invariant norm of p. The two possibilities are distinguished by how they transform. In particular, the square norm of p transforms as

$$m^2 = p\bar{p} \rightarrow \left(LpL^\dagger\right)\left(\bar{L}^\dagger \bar{p}\bar{L}\right) = p\bar{p} \tag{5}$$

whereas the rest-frame momentum becomes

$$me_0 \rightarrow Lme_0L^\dagger = mLL^\dagger. \tag{6}$$

The extension from $C\ell_3$ to $C\ell_7$ requires four additional basis vectors, e_4, e_5, e_6, e_7, that are orthogonal to physical space, namely the span of $\{e_1, e_2, e_3\}$, which generates the $C\ell_3$ considered here. If z is any linear combination of e_4, e_5, e_6, e_7, its product with any $K \in C\ell_3$ satisfies

$$zK = \bar{K}^\dagger z. \tag{7}$$

It follows that z is invariant under any Lorentz transformation (4) with $L \in C\ell_3$: $z \rightarrow LzL^\dagger = L\bar{L}z = z$. More general rotations in $C\ell_7$ have the form of equation (4) but are generated by bivectors that are not restricted to the three spatial planes of $C\ell_3$.

It is natural to question the significance of the extra dimensions. Of course they may be compact as in the Kaluza–Klein approach, but then one can still perform rotations in the tangent space at any point, for example in the e_1e_4 plane or in other planes involving the extra dimensions. Alternatively, the extra dimensions may be finite or infinite in extent but simply not observable as spatial degrees of freedom. One way to arrive at $C\ell_7$ from $C\ell_3$ is to seek a metric-free foundation for $C\ell_3$. The anticommutation relation (1) implies a Euclidean spatial metric, but we may instead start with a three-dimensional metric-free Witt basis [19, 20] of null vectors $\{\alpha_1, \alpha_2, \alpha_3\}$ satisfying

$$\alpha_j\alpha_k + \alpha_k\alpha_j = 0 \qquad j, k = 1, 2, 3. \tag{8}$$

A dual space can then be defined as the span of $\left\{\alpha_1^*, \alpha_2^*, \alpha_3^*\right\}$, where

$$\alpha_j^*\alpha_k + \alpha_k\alpha_j^* = \delta_{jk}. \tag{9}$$

The anticommutation relation (1) for $C\ell_3$ follows directly from the identification $e_k = \alpha_k + \alpha_k^*$. However, there are now three extra linearly independent vectors that we can label $e_{-k} = \alpha_k - \alpha_k^*$. It is easily verified that the six basis vectors $e_{\pm k}$ anticommute and square to ± 1. The span of $\{e_{\pm k}\}_{1\leqslant k\leqslant 3}$ is a six-dimensional space with the metric signature (3, 3). It generates the Clifford algebra $C\ell_{3,3}$, and its volume element $e_4 \equiv e_{-3}e_{-2}e_{-1}e_1e_2e_3$ squares to +1 and anticommutes with the six $e_{\pm k}$. As in the familiar Dirac algebra, the volume element in $C\ell_{3,3}$ acts as an additional spatial dimension. It can be added to the basis to form a seven-dimensional space with the corresponding universal Clifford algebra $C\ell_{4,3}$. The algebra $C\ell_{4,3}$ can be mapped to $C\ell_7$ if we assume the existence of a scalar unit imaginary element i. We replace the three e_{-k} by elements $e_{4+k} \equiv \mathrm{i}e_{-k}$ that square to +1. The elements e_j, with $j = 1, 2, \ldots, 7$, then satisfy equations (1) and (2) and span a seven-dimensional Euclidean space such as used here. The Witt basis elements can now be written

$$\alpha_k = \tfrac{1}{2}\,(e_k - \mathrm{i}e_{4+k}) \qquad k = 1, 2, 3 \tag{10}$$

and if we take the e_j to be Hermitian, the dual elements are their Hermitian conjugates: $\alpha_k^* = \alpha_k^\dagger$. The anticommutation relations (8), (9) are just those of fermion annihilation and creation operators, whose products, together with other constructions analogous to equation (10), can generate the isotopic groups used below. Here we derive the group generators directly in terms of bivectors of $C\ell_7$ by demanding that they avoid interactions with the right-chiral neutrino and leave the spacetime components of the particle and antiparticle currents invariant.

To illustrate relations in $C\ell_7$, it is useful to have an explicit matrix representation. Such a representation can be built from a 4×4-matrix representation of the familiar Dirac algebra $C\ell_{1,3}$, in which the basis vectors satisfy $\gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu = 2\eta_{\mu\nu}$. However, we note that the imaginary unit is not part of $C\ell_{1,3}$, that the volume element $\mathrm{i}\gamma_5 = \gamma_0\gamma_1\gamma_2\gamma_3$ plays the role of an added spatial dimension, that the γ_μ cannot all be Hermitian and that γ_0 has additional significance in the definition of the conjugate spinor. A faithful 8×8 matrix representation of $C\ell_7$ can be expressed in the block-matrix form

$$\begin{aligned}
&e_0 = 1 \leftrightarrow \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad e_k \leftrightarrow \begin{pmatrix} -\gamma_0\gamma_k & 0 \\ 0 & -\gamma_0\gamma_k \end{pmatrix} \qquad e_4 \leftrightarrow \begin{pmatrix} \mathrm{i}\gamma_0\gamma_5 & 0 \\ 0 & \mathrm{i}\gamma_0\gamma_5 \end{pmatrix} \\
&e_5 \leftrightarrow \begin{pmatrix} \gamma_0 & 0 \\ 0 & -\gamma_0 \end{pmatrix} \qquad e_6 \leftrightarrow \begin{pmatrix} 0 & \gamma_0 \\ \gamma_0 & 0 \end{pmatrix} \qquad e_7 \leftrightarrow \begin{pmatrix} 0 & -\mathrm{i}\gamma_0 \\ \mathrm{i}\gamma_0 & 0 \end{pmatrix}
\end{aligned} \tag{11}$$

with $k = 1, 2, 3$. Each basis vector e_j is thus represented by a Hermitian matrix. It can be seen that this representation absorbs γ_0 into the definition of a spatial direction, thus relegating time to the scalar part of the algebra, and it introduces four extra spacelike dimensions in accordance with the defining anticommutator (1) so that $[\mathrm{i}]_{8\times 8}$ arises naturally through the full volume element. Operations involving these higher dimensions may now be stated and executed cleanly in terms of the basis vectors e_j without having to appeal to products of gamma matrices. The representation (11) is only one of many that absorb the Dirac algebra into the more mathematically uniform $C\ell_7$. In fact, the model can be presented algebraically without reference to specific matrices, but the representation (11) is useful for understanding the spinorial element and for making comparisons to conventional expressions.

3. Algebraic spinors and currents

Algebraic spinors may be defined as entities that transform under the restricted Lorentz group not as vectors (4), but according to the rule

$$\Psi \to L\Psi. \tag{12}$$

They obey a similar transformation law under translations. Spinors are thus elements of the carrier space of a representation (generally a reducible representation) of the Poincaré group. In the $C\ell_3$ version of the Dirac theory [15], the spinor field Ψ, represented by a 2×2 matrix, is an amplitude of the bilinear Lorentz transformation (4) relating the reference and laboratory frames of the particle. The current, in particular, corresponds to the transformation of the rest-frame time axis: $J = \Psi\Psi^\dagger$.

To describe one generation of the standard model, we use the algebraic spinor $\Psi \in C\ell_7$. It is represented by an 8×8 matrix whose columns contain the spinors for the leptons (ν, e) as well as for three colors of quarks (u, d) and all their antiparticles. Presumably it specifies not only the motion and orientation of the particles in spacetime, but also in the space spanned by the extra four dimensions.

The transformations (12) are preserved by multiplication from the right by Lorentz-invariant factors, in particular by Hermitian idempotent elements (projectors) that project the spinor Ψ onto left ideals of $C\ell_7$. In particular, there are eight independent primitive idempotents that, in terms of matrices, can each be used to reduce Ψ to a single nonvanishing column (examples are given below and in the appendix). Every column of Ψ transforms as a spinor (equation (12)), and operators from the left do not mix the spinors in different columns.

In constructing an algebraic expression for the particle current J, we seek a form that is bilinear in the spinors, transforms as a vector, and satisfies $J^\dagger = J$. The last requirement ensures that the physical components J^μ of J are real. The simplest solution to this, and the one that we adopt here, has the same form as found for the Dirac theory in $C\ell_3$:

$$J = \Psi\Psi^\dagger \to L\Psi\Psi^\dagger L^\dagger. \tag{13}$$

A specific component of J may be extracted by contracting it with its associated direction through

$$J_\mu = \langle \Psi\Psi^\dagger \bar{e}_\mu \rangle_S = \langle \Psi^\dagger \bar{e}_\mu \Psi \rangle_S. \tag{14}$$

(We have used the algebraic property $\langle AB\rangle_S = \langle BA\rangle_S$, whose matrix representation through $\langle \cdots \rangle_S \leftrightarrow \frac{1}{8}\,\mathrm{tr}(\cdots)$ is the familiar trace theorem $\mathrm{tr}(AB) = \mathrm{tr}(BA)$.) From the matrix representation for e_μ, we see that the components (14) are sums of the conventional expressions $\lceil \bar{\psi}\gamma_\mu\psi \rfloor$ for each fermion and antifermion, where the delimiters $\lceil \cdots \rfloor$ designate prevailing non-algebraic notation.

It is useful to distinguish transformations acting on the left from others that act on the right. Those on the left include Lorentz transformations and rotations in the space of the extra four dimensions. Since they operate on orthogonal subspaces, rotations in the space spanned by $\{e_4, e_5, e_6, e_7\}$ commute with the Lorentz transformations. They are applied to the spinor after the particles have been given the motion and orientation described by Ψ and will be called 'exterior' transformations to represent their position, as in equation (13), in transformations of the current J. Transformations applied from the right will similarly be called 'interior'. They are applied to the particles in their reference frame, before they acquire the motion and orientation implied by the spinor. Note that exterior transformations are not synonymous with *external* transformations, since the extra four dimensions may relate to properties that are commonly considered to be *internal*. Exterior transformations mix the components within a single pair of fermions, whereas interior transformations mix different pairs together.

Primitive idempotents $P(n)$ needed to isolate columns of Ψ can be constructed from interior products of three pairs of simple projectors $P_\pm = P_\pm^\dagger = P_\pm^2 = \bar{P}_\mp$, where $P_+ + P_- = 1$ and $P_+P_- = 0$. From among several equivalent choices, we use the three mutually commuting projector pairs

$$P_{\pm 3} \equiv \tfrac{1}{2}(1 \pm e_3) \qquad P_{\pm\alpha} \equiv \tfrac{1}{2}(1 \pm \mathrm{i}e_4e_5) \qquad P_{\pm\beta} \equiv \tfrac{1}{2}(1 \pm \mathrm{i}e_6e_7). \tag{15}$$

Table 1. The algebraic P_{+3} (particle) spinors, where the two-component Weyl spinors are *algebraic* elements defined by $\psi_{\rm R} = \psi_0 + \psi_1 e_1$, and $\psi_{\rm L} = \psi_3 - \psi_2 e_1$ for each particle with Dirac spinor components $\psi_0, \psi_1, \psi_2, \psi_3$.

Lower spinor	Upper spinor
$P_{+\beta}\Psi P(1) = \sqrt{8}(\psi_{\rm R}e_6e_5 + \psi_{\rm L}e_1e_6)P(1)$	$P_{-\beta}\Psi P(1) = \sqrt{8}(\psi_{\rm R} + \psi_{\rm L}e_1e_5)P(1)$
$P_{+\beta}\Psi P(4) = \sqrt{8}(\psi_{\rm R}e_6e_1 + \psi_{\rm L}e_6e_5)P(4)$	$P_{-\beta}\Psi P(4) = \sqrt{8}(\psi_{\rm R}e_5e_1 + \psi_{\rm L})P(4)$
$P_{+\beta}\Psi P(5) = \sqrt{8}(\psi_{\rm R} + \psi_{\rm L}e_5e_1)P(5)$	$P_{-\beta}\Psi P(5) = \sqrt{8}(\psi_{\rm R}e_5e_6 + \psi_{\rm L}e_1e_6)P(5)$
$P_{+\beta}\Psi P(8) = \sqrt{8}(\psi_{\rm R}e_1e_5 + \psi_{\rm L})P(8)$	$P_{-\beta}\Psi P(8) = \sqrt{8}(\psi_{\rm R}e_6e_1 + \psi_{\rm L}e_5e_6)P(8)$

In the Weyl γ-matrix representation adopted here (see appendix), the products $P_{\pm3}P_{\pm\alpha}P_{\pm\beta}$ are simply the eight diagonal matrices with a single nonvanishing, unit element. For example, $P_{+3}P_{+\alpha}P_{-\beta} = \mathrm{diag}[1,0,0,0,0,0,0,0] \equiv P(1)$, and the first-column spinor may be written $\Psi P(1)$ (see appendix). Each of the eight primitive projectors $P(n)$, applied from the right, projects Ψ (or other elements) onto one of eight minimal left ideals of $C\ell_7$ and one of the eight columns of the matrix representation. The nth column $\Psi P(n)$ is identified with a distinct pair of fermions and forms current elements in equation (13) only with itself[1].

One pair of simple projectors, applied from the right, can be taken to separate particles from antiparticles. We let this be $P_{\pm3}$, although this choice is generalized below. Thus, columns $1, 4, 5, 8$, selected by P_{+3}, are designated for particles and the remaining columns, selected by P_{-3}, contain the antiparticle spinors. Each column holds the spinors for a fermion doublet, and the projectors for the two isotopic-spin components are taken to be $P_{\pm\beta}$ applied as an *exterior* operator (from the left). In the Weyl representation [8], each four-component spinor in Ψ is further split into two-component spinors of right and left chirality. For example, the upper spinor of column one comprises the nonvanishing components of $P_{-\beta}\Psi P(1)$:

$$\begin{pmatrix}\Psi_{11}\\ \Psi_{21}\\ \Psi_{31}\\ \Psi_{41}\end{pmatrix} \equiv \sqrt{8}\begin{pmatrix}\psi_0\\ \psi_1\\ \psi_2\\ \psi_3\end{pmatrix} = \sqrt{8}\left\lceil \begin{pmatrix}\psi_{\rm R}\\ \psi_{\rm L}\end{pmatrix}\right\rfloor \tag{16}$$

where $\sqrt{8}$ factors are inserted to agree with conventional normalization. The lower spinor $P_{+\beta}\Psi P\ (1)$ with the four nonzero components Ψ_{51} to Ψ_{81} and the other P_{+3} columns are labelled in a similar manner. The P_{+3} (particle) spinors can be factored explicitly as in table 1. The P_{-3} spinors have a similar form but have been excluded for brevity. Indeed, one need only work out the algebraic equivalent of the first column, since the remaining P_{+3} columns are easily obtained by multiplying the first-column spinor from the right by the elements e_5e_1, e_5e_6, e_6e_1, which shifts it to columns 4, 5, 8 respectively. These algebraic spinors transform under $\Psi \to L\Psi$ in the same manner as in the conventional column representation.

The chiral projectors for all fermions in the Weyl representation are the mutually annihilating exterior operators

$$P_{\rm R/L} = P_{\bar{\rm L}/\bar{\rm R}} = \tfrac{1}{2}(1 \pm e_4e_5e_6e_7). \tag{17}$$

By the 'pacwoman' [4] property $P_{\rm R/L} = \pm e_4e_5e_6e_7P_{\rm R/L} = \mp \mathrm{i}e_1e_2e_3P_{\rm R/L}$, these projectors split $C\ell_7$ into parts in which the basis elements e_1, e_2, e_3 of physical space have right and

[1] A similar primitive-idempotent structure for particle doublets was proposed for the algebra $C\ell_{1,6}$ by Chisholm and Farwell [11, 13]. However, in spite of an isomorphism between $C\ell_7$ and $C\ell_{1,6}$, their restriction to spinors belonging to minmal left ideals allows them to include only one isotopic pair of particles whereas our spinor contains eight isotopic pairs of particles and antiparticles. Furthermore, our use of paravectors provides additional degrees of freedom. Indeed, it corresponds to a two-to-one mapping of the larger $C\ell_{1,7}$ onto its even subalgebra $C\ell^+_{1,7} \simeq C\ell_7$.

Table 2. Column designations of the matrix representation of Ψ in terms of common two-component chiral spinors.

ℓ	$-\bar{q}_{\mathrm{grn}}$	$\bar{q}_{\mathrm{blu}}$	q_{red}	q_{grn}	$\bar{\ell}$	$-\bar{q}_{\mathrm{red}}$	q_{blu}
ν_{R}	$\bar{d}_{\mathrm{L}}$	$-\bar{d}_{\mathrm{L}}$	u_{R}	u_{R}	$-\bar{e}_{\mathrm{L}}$	$\bar{d}_{\mathrm{L}}$	u_{R}
ν_{L}	$\bar{d}_{\mathrm{R}}$	$-\bar{d}_{\mathrm{R}}$	u_{L}	u_{L}	$-\bar{e}_{\mathrm{R}}$	$\bar{d}_{\mathrm{R}}$	u_{L}
e_{R}	$-\bar{u}_{\mathrm{L}}$	$\bar{u}_{\mathrm{L}}$	d_{R}	d_{R}	$\bar{\nu}_{\mathrm{L}}$	$-\bar{u}_{\mathrm{L}}$	d_{R}
e_{L}	$-\bar{u}_{\mathrm{R}}$	$\bar{u}_{\mathrm{R}}$	d_{L}	d_{L}	$\bar{\nu}_{\mathrm{R}}$	$-\bar{u}_{\mathrm{R}}$	d_{L}

left-handed orientations, respectively. In particular, since

$$e_1e_2e_3P_{\mathrm{R/L}} = \pm \mathrm{i} P_{\mathrm{R/L}} \tag{18}$$

the spatial volume element $e_1e_2e_3$ (which is equal to the spacetime volume element $e_0e_1e_2e_3$ in the laboratory frame) can be replaced[2] by +i when multiplying P_{R} and $-\mathrm{i}$ when multiplying P_{L}. Note that the chirality projectors $P_{\mathrm{R/L}}$ commute with all elements of the subalgebra $C\ell_3$ as well as with $P_{\pm 3}$, $P_{\pm\alpha}$, $P_{\pm\beta}$ and therefore with all the primitive idempotents $P(n)$. Furthermore, any element x of $C\ell_7$ with an odd number of vector factors from the higher dimensions e_4, e_5, e_6, e_7 reverses the chirality: $xP_{\mathrm{R}} = P_{\mathrm{L}}x$. Such elements include bivectors such as e_3e_4 that can generate rotations from a left-handed coordinate system into a right-handed one and *vice versa*. The chirality of Ψ can thus be flipped by the transformation

$$\Psi \to -e_1e_2e_3e_4\Psi \tag{19}$$

which has the effect of reversing the vector components of the current (13) in the span of $\{e_1, e_2, e_3, e_4\}$ while leaving the components in the span of $\{e_0, e_5, e_6, e_7\}$ invariant.

Charge conjugation is realized by the algebraic operation

$$\Psi \to \Psi_{\mathrm{C}} = \mathrm{i}e_4\overline{\Psi}^{\dagger}. \tag{20}$$

The combination of the two antiautomorphic involutions obeys the rule $\overline{(AB)}^{\dagger} = \bar{A}^{\dagger}\bar{B}^{\dagger}$, and the conjugate of the upper spinor of the first column (see table 1), for example, is

$$P_{+\beta}\Psi_{\mathrm{C}}P(6) = \mathrm{i}e_4\sqrt{8}(\bar{\psi}_{\mathrm{R}}^{\dagger} + \bar{\psi}_{\mathrm{L}}^{\dagger}e_1e_5)P(6) \tag{21}$$

where $\bar{\psi}_{\mathrm{R}}^{\dagger} = \psi_0^* - \psi_1^*e_1$ and $\bar{\psi}_{\mathrm{L}}^{\dagger} = \psi_3^* + \psi_2^*e_1$. The identification (20), together with the relation (7) and transformation rule (12), ensures that spinors and their charge conjugates transform in the same way under the Lorentz group:

$$\Psi_{\mathrm{C}} \to \mathrm{i}e_4\bar{L}^{\dagger}\bar{\Psi}^{\dagger} = L\mathrm{i}e_4\bar{\Psi}^{\dagger} = L\Psi_{\mathrm{C}}. \tag{22}$$

In the matrix representation, charge conjugation (20) is equivalent to defining the conventional charge conjugates through $\lceil \psi_{\mathrm{C}} = \mathrm{i}\gamma^2\psi^* \rfloor$ and interchanging both corresponding particle and antiparticle columns and upper and lower spinors. The resulting full structure of Ψ is shown in table 2. For the sake of brevity, we take the liberty of labelling the spinors with the particle designations shown, although the gauge structure has not yet been determined. This is one of many possible arrangements and will be generalized below. Note that charge conjugation reverses the signs on all of the simple interior and exterior projectors used here.

Geometrically, charge conjugation transforms the particle current as

$$J = \Psi\Psi^{\dagger} \to e_4\bar{\Psi}^{\dagger}\bar{\Psi}e_4 = e_4\bar{J}e_4 \tag{23}$$

and has the effect of negating the e_4 component while leaving all other directions invariant. This is a discrete symmetry of the higher-dimensional directions that is not accessible by a

[2] A potential conflict between these cases is restricted by the association of observed quantities such as J_μ (14) with the scalar expressions.

rotation. The negation of two or four directions can be achieved by rotations, and to negate three directions one simply reverses one direction followed by reversing another two or four. The choice of e_4 and the phase introduced in equation (20) are merely convenient choices for the representation used.

The total current obtained by simply adding all the left ideal doublets into a single element Ψ is then

$$J \equiv \Psi\Psi^\dagger = \sum_{a=1}^{16} \lceil J^\mu_{(a)} \rfloor e_\mu + \text{(higher-dim. terms)}. \tag{24}$$

The sum here runs over the 16 four-component spinors assigned to the upper and lower halves of the eight minimal left ideals, each of which is ascribed to a distinct fermion [13]. The residual part of the current involves cross-current terms between the upper and lower fermions of the same ideal as well as masslike terms of the form $\lceil \bar{\psi}\psi \rfloor$, all projected onto higher-dimensional elements.

The main idea of this section has simply been that, instead of writing a separate term for each of the particle currents, we can consolidate them into a single expression that accommodates a number of spinorial representations. The advantage of the algebraic formalism becomes evident when we enumerate all the possible rotational symmetries of this current.

4. Gauge symmetries

The algebraic current (13) holds all the chiral currents of a single generation of the standard model, with distinct antiparticle currents, as a generalized current in a linear space of seven spatial dimensions. In this section, we show that rotational transformations that leave both the physical spacetime components of the particle and antiparticle currents and the right-chiral neutrino (and left-chiral antineutrino) invariant lead exactly to the standard-model gauge symmetries. Our approach is analogous to the conventional case where one notices that $\lceil \psi \to \exp(\mathrm{i}\theta)\psi \rfloor$ is a symmetry of the current, but now we consider all possible rotations in the seven-dimensional Euclidean space. This involves generators acting from both the left and right of the algebraic spinor, as these generators usually do not commute with Ψ. We show further that rotations are the only continuous transformations acting from either the right or the left that are allowed in our model. Thus, by combining the fermion currents into the single form (13), we uncover relationships among the fermions that in most other models are simply imposed on abstract spaces.

We begin by considering exterior rotations $\Psi \to G\Psi = \exp(\theta T)\Psi$ that leave the physical spacetime components of $\Psi\Psi^\dagger$ invariant, where T generates rotations in one or more planes of the seven-dimensional space. As seen above, the generator of rotations in a plane is the bivector for the plane, and bivectors are anti-Hermitian. From the infinitesimal form

$$J \to (1 + \theta T)\Psi\Psi^\dagger(1 + \theta T^\dagger) \tag{25}$$

it is clear from the invariance of J_μ (14) for $\mu = 0, 1, 2, 3$, that $e_\mu T = -T^\dagger e_\mu = T e_\mu$. Thus, to leave the spatial components of J invariant, T must commute with e_k, $k = 1, 2, 3$. This reduces the choices for T to linear combinations of the six bivectors $e_j e_k : (j, k) \in \{4, 5, 6, 7\},\ j > k$, of the Lie algebra $so(4)$, which generate rotations of the higher-dimensional vector components of the current among themselves. As seen above, generators formed from products of e_4, e_5, e_6, e_7 are invariant under Lorentz transformations and may therefore be associated with intrinsic transformations. The projectors $P_{\mathrm{R/L}}$ split $so\,(4)$ into two independent copies of the algebra $su\,(2)$, corresponding to the rotation groups $SU\,(2)_{\mathrm{L/R}}$ with generators of the form $e_j e_k P_{\mathrm{L/R}}$.

The generators of $SU\,(2)_{\mathrm{L}}$ may be written in the form

$$T_1 = \tfrac{1}{4}(e_6e_4 + e_5e_7) \qquad T_2 = \tfrac{1}{4}(e_7e_4 + e_6e_5) \qquad T_3 = \tfrac{1}{4}(e_5e_4 + e_7e_6) \quad (26)$$

that implicitly contains the left-chiral projector (17), for example $2T_1 = e_6e_4P_{\mathrm{L}}$, and therefore acts only on left-chiral particles and right-chiral antiparticles. The three generators (26) induce simultaneous rotations in a pair of commuting planes and satisfy $[T_a, T_b] = \varepsilon_{abc}T_c$, with the fully antisymmetric structure constants ε_{abc} where $\varepsilon_{123} = 1$. The conventional presence of the imaginary unit in front of T_c has been absorbed into the anti-Hermitian property of the bivectors. The effect of the transformation $\Psi \to \exp(\theta_a T_a)\Psi$ is identical to that of the prevailing $SU(2)$ prescriptions

$$\begin{pmatrix} \nu_{\mathrm{L}} & u_{\mathrm{L}} & -\bar{e}_{\mathrm{R}} & -\bar{d}_{\mathrm{R}} \\ e_{\mathrm{L}} & d_{\mathrm{L}} & \bar{\nu}_{\mathrm{R}} & \bar{u}_{\mathrm{R}} \end{pmatrix} \to \exp(-\mathrm{i}\theta_a\sigma_a/2) \begin{pmatrix} \nu_{\mathrm{L}} & u_{\mathrm{L}} & -\bar{e}_{\mathrm{R}} & -\bar{d}_{\mathrm{R}} \\ e_{\mathrm{L}} & d_{\mathrm{L}} & \bar{\nu}_{\mathrm{R}} & \bar{u}_{\mathrm{R}} \end{pmatrix} \quad (27)$$

as is readily verified by computing the matrix representations of the generators. Because operations from the left shuffle entire rows about in the matrix representation but do not shift columns, the assignment of doublets to columns is still arbitrary. The three linearly independent generators formed by replacing the + signs in (26) by − signs, and indeed any linear combination of them, all have the form $x_b P_{\mathrm{R}}$, where x_b is a bivector. They would thus couple with ν_{R} and its conjugate and are therefore omitted.

Now let us look at the possible *interior* rotations $\Psi \to \Psi G' = \Psi\exp(\theta T')$. To emphasize the fact that they act on the *right* side of Ψ, the interior transformations and generators are denoted here with a prime. Any interior unitary transformation leaves $J = \Psi\Psi^\dagger$ invariant, but we want a stronger condition: we demand that the spacetime components of the particle and antiparticle currents be separately invariant. Mathematically, this is equivalent to splitting the current in two using the $\Psi P_{\pm 3}$ spinors

$$J = \tfrac{1}{2}\Psi(1 + e_3)\Psi^\dagger + \tfrac{1}{2}\Psi(1 - e_3)\Psi^\dagger \equiv J_{+3} + J_{-3} \quad (28)$$

and requiring each part to be invariant. Recall that P_{+3} and P_{-3} are projectors for particles and antiparticles, respectively, and remember that the interior projectors do not Lorentz transform; they represent a choice in the intrinsic or reference-frame structure of the particles and are not altered by a Lorentz transformation operating from the opposite side of the spinors. Generators acting between Ψ and $\Psi^\dagger$ are similarly Lorentz invariant. Thus, we may involve the elements e_1, e_2, e_3 in the interior symmetries while satisfying the Coleman–Mandula theorem [6], which prohibits any non-trivial combination of the Poincaré and isotopic groups. Under the infinitesimal interior transformation $\Psi \to \Psi(1 + \theta T')$, we have

$$J_{\pm 3} \to \tfrac{1}{2}\Psi(1 + \theta T')(1 \pm e_3)(1 + \theta T'^\dagger)\Psi^\dagger \quad (29)$$

which may be viewed as a transformation of the central $P_{\pm 3}$ projector. We see that the space of available bivector generators that leave e_3 invariant is now spanned by the larger set of 15 bivectors $e_je_k : (j, k) \in \{1, 2, 4, 5, 6, 7\},\ \ j < k$. Insulating the right-chiral neutrino from interior transformations in a similar manner as before now requires that both lepton columns (1 and 6 in the representation adopted) be avoided. This reduces the number of independent generators to eight, all of which couple quarks of different colour charges:

$$\begin{array}{lll} T'_1 = \tfrac{1}{4}(e_1e_7 + e_6e_2) & T'_2 = \tfrac{1}{4}(e_1e_6 + e_2e_7) & T'_3 = \tfrac{1}{4}(e_1e_2 + e_7e_6) \\ T'_4 = \tfrac{1}{4}(e_6e_4 + e_5e_7) & T'_5 = \tfrac{1}{4}(e_4e_7 + e_5e_6) & T'_6 = \tfrac{1}{4}(e_4e_1 + e_2e_5) \\ T'_7 = \tfrac{1}{4}(e_1e_5 + e_2e_4) & T'_8 = \dfrac{1}{4\sqrt{3}}(e_2e_1 + 2e_5e_4 + e_7e_6). & \end{array} \quad (30)$$

The interior generators have been arranged to give the conventional $SU(3)$ structure constants [8]

$$[T'_a, T'_b] = -f_{abc}T'_c. \tag{31}$$

Computing the matrix representation for each of these generators using (11), we find that the transformation $\Psi \to \Psi \exp(\theta_a T'_a)$ is identical in its effect on the P_{+3} spinor components to

$$\left(q_{\mathrm{red}}, q_{\mathrm{grn}}, q_{\mathrm{blu}}\right) \to \left(q_{\mathrm{red}}, q_{\mathrm{grn}}, q_{\mathrm{blu}}\right)\exp(-\mathrm{i}\theta_a\lambda_a^*/2) \tag{32}$$

where λ_a are the Gell-Mann matrices. This is equivalent to the more familiar

$$\begin{pmatrix} q_{\mathrm{red}} \\ q_{\mathrm{grn}} \\ q_{\mathrm{blu}} \end{pmatrix} \to \exp(-\mathrm{i}\theta_a\lambda_a/2)\begin{pmatrix} q_{\mathrm{red}} \\ q_{\mathrm{grn}} \\ q_{\mathrm{blu}} \end{pmatrix}. \tag{33}$$

Under the same algebraic operation, the effect of the remaining submatrices on the conjugate spinors $(-\overline{q_{\mathrm{grn}}}, \overline{q_{\mathrm{blu}}}, -\overline{q_{\mathrm{red}}})$ is equivalent to

$$\left(\overline{q_{\mathrm{red}}}, \overline{q_{\mathrm{grn}}}, \overline{q_{\mathrm{blu}}}\right) \to \left(\overline{q_{\mathrm{red}}}, \overline{q_{\mathrm{grn}}}, \overline{q_{\mathrm{blu}}}\right)\exp(\mathrm{i}\theta_a\lambda_a/2) \tag{34}$$

which is the correct transformation. The fact that the doublets can be written in the same representation by using either the column (u, d) or the column $(-\bar{d}, \bar{u})$ is a special property of $SU(2)$. Such a construction is not possible for the $SU(3)$ triplet, but the geometric symmetries here provide a separate set of $SU(3)$ submatrices, one in terms of $-\lambda_a^*$ and the other in terms of λ_a, operating on the two carrier spaces. It is an advantage of having the conjugate spinors in separate columns of Ψ that the same algebraic symmetry applies to both particles and antiparticles.

Since any operation from the left shuffles rows whereas one from the right shuffles columns, the order in which two such operations is applied is immaterial. Therefore, *it is of no consequence that the generators from the left do not necessarily commute with the generators acting from the right.* They act on independent structural elements (rows and columns) of Ψ and thus effect transformations as if they were two commuting symmetries in an abstract space. This property, together with the higher dimensionality of the linear subspace of bivectors, is basically how these gauge groups arise from only four extra dimensions.

There remains one additional possible symmetry. We need to consider a synchronized double-sided rotation that conspires to cancel out in the case of the right-chiral neutrino. As this rotation is to represent a distinct symmetry, its left- and right-side generators must commute with all $SU(2)$ and $SU(3)$ generators, respectively. Since both the right- and left-sided parts separately couple the right-chiral neutrino, we resurrect previously discarded generators. The surviving bivector candidates are $(e_4e_5 + e_7e_6)$ acting from the left, and $(e_1e_2 + e_5e_4 + e_6e_7)$ operating from the right. One may verify with the infinitesimal operator

$$\Psi \to (1 + \theta_0 T_0)\Psi(1 + \theta_0 T'_0) \tag{35}$$

that the solution for which there is no change to the right-chiral neutrino can be normalized to

$$T_0 = \tfrac{1}{2}(e_4e_5 + e_7e_6) \qquad T'_0 = \tfrac{1}{3}(e_1e_2 + e_5e_4 + e_6e_7). \tag{36}$$

Applying this operation to each spinor in turn proves to be identical to the $U(1)_Y$ transformation $\psi_{(j)} \to \exp(-\mathrm{i}\theta_0 Y_{(j)})\psi_{(j)}$ with the weak hypercharge assignments

$$\begin{aligned} Y(\nu_{\mathrm{R}}, \nu_{\mathrm{L}}, e_{\mathrm{R}}, e_{\mathrm{L}}) &= (0, -1, -2, -1) = -Y(\bar{\nu}_{\mathrm{L}}, \bar{\nu}_{\mathrm{R}}, \bar{e}_{\mathrm{L}}, \bar{e}_{\mathrm{R}}) \\ Y(u_{\mathrm{R}}, u_{\mathrm{L}}, d_{\mathrm{R}}, d_{\mathrm{L}}) &= (4/3, 1/3, -2/3, 1/3) = -Y(\bar{u}_{\mathrm{L}}, \bar{u}_{\mathrm{R}}, \bar{d}_{\mathrm{L}}, \bar{d}_{\mathrm{R}}). \end{aligned} \tag{37}$$

It produces the conventional weak hypercharge assignments for both leptons and quarks.

The above transformations may now be combined into a single expression

$$\Psi \to \exp(\theta_0 T_0 + \theta_a T_a)\Psi \exp(\theta_0 T_0' + \theta_b' T_b') \tag{38}$$

operating on both particles and antiparticles. This exhausts the rotational gauge symmetries. The double-sided transformations may be locally gauged by introducing 12 gauge fields $B, W_a, G_a \in C\ell_3$ that transform according to

$$\begin{aligned}
\bar{B} &\to \bar{B} + \frac{2}{g'}\bar{\partial}\theta_0 \\
\bar{W}_a &\to \bar{W}_a + \frac{1}{g}\bar{\partial}\theta_a + \varepsilon_{abc}\theta_b\bar{W}_c \qquad && a \in \{1, 2, 3\} \\
\bar{G}_a &\to \bar{G}_a + \frac{1}{g_s}\bar{\partial}\theta_a' + f_{abc}\theta_b'\bar{G}_c \qquad && a \in \{1, 2, \ldots, 8\}
\end{aligned} \tag{39}$$

into the Lagrangian derivative terms

$$\mathcal{L}_\partial = \langle\Psi^\dagger \mathrm{i}\bar{\partial}\Psi\rangle_\mathrm{S} - \frac{g'}{2}\langle\Psi^\dagger \mathrm{i}\bar{B}(T_0\Psi + \Psi T_0')\rangle_\mathrm{S} - g\langle\Psi^\dagger \mathrm{i}\bar{W}_a T_a\Psi\rangle_\mathrm{S} - g_s\langle\Psi^\dagger \mathrm{i}\bar{G}_a\Psi T_a'\rangle_\mathrm{S} \tag{40}$$

where the algebraic derivative operator is defined by [4]

$$\bar{\partial} \equiv \partial_0 + \partial_1 e_1 + \partial_2 e_2 + \partial_3 e_3. \tag{41}$$

When used with the interior and exterior generators found above, expression (40) yields all the usual particle and antiparticle charge currents. Note that all bivector generators uniformly obey $T^\dagger = \bar{T} = -T$, and all exterior T commute with the physical gauge fields. Although the above terms are similar to the conventional forms, it should be emphasized that all of the currents are simultaneously handled in the same expression using the algebraic spinor Ψ, whose gauge symmetries arise naturally from the geometry of the model.

It is of interest to relax the condition that the transformations of Ψ be rotations and to see whether generators other than bivectors might play a role. However, the unitarity of the transformations together with the consistency of charge conjugation (20) combine with the invariance of spacetime components of the particle and antiparticle currents and the sterility of ν_R to restrict both exterior and interior generators to bivectors. Explicitly, unitarity requires T and T' to be anti-Hermitian ($T = -T^\dagger$), restricting them to real linear combinations of products of two, three, six or seven vectors. Consistency requires charge conjugation to commute with the interior and exterior transformations, yielding $Te_4 = e_4\bar{T}^\dagger$ and $T' = \bar{T}'^\dagger$. These relations eliminate all odd elements except trivectors of T that anticommute with e_4. The invariance of the e_1, e_2, and e_3 components of J further eliminates six-vectors as well as all trivectors except $e_1e_2e_3$ from admissible contributions to T. The separation of particle and antiparticle currents requires T' to commute with e_3, which reduces the possible contributions to T' to bivectors plus the one six-vector ie_3. The only remaining candidates that are not bivectors are thus $e_1e_2e_3$ for T and ie_3 for T', and both of these can be eliminated because of their coupling to ν_R. Thus, even after relaxing the condition that the transformations be rotations, we find that the generators of the interior and exterior transformations must be bivectors. Furthermore, as seen above, in order to avoid ν_R, the exterior generators must belong to the left ideal in which $T = TP_\mathrm{L}$, where P_L is the simple chiral projector. From the form (17) of P_L, we see that the generators T are linear combinations of pairs of commuting bivectors, generating simultaneous rotations in orthogonal planes. Similarly, the interior generators belong to the right ideal in which $T' = P_{q\bar{q}}T'$ where $P_{q\bar{q}} = 1 - P(1) - P(6)$ (see the appendix) is the quark–antiquark projector, which can also be expressed as a linear combination of simple projectors. The generators T' are thus also seen to be linear combinations of pairs of commuting bivectors.

A similar relaxation of the rotation requirement for the synchronized double-sided transformation (35) leaves T_0 unchanged but adds a six-vector term to T_0' (36):

$$T_0' = \beta(e_1e_2 + e_5e_4 + e_6e_7) + (1 - 3\beta)\mathrm{i}e_3. \tag{42}$$

By restricting the possible transformations to rotations in the seven-dimensional space, we have effectively chosen $\beta = 1/3$.

The $U(1)_Y \otimes SU(2)_\mathrm{L} \otimes SU(3)_\mathrm{C}$ result here is a general consequence of the algebra for rotations that conserve the particle and antiparticle currents and do not couple ν_R. They are not specific to the ΨP_{+3} spinors. Any arbitrary fitting of the doublets into some orthogonal linear combination of the columns is accessible by shuffling the P_{+3} columns through a transformation $\Psi \to \Psi S$ where $SS^\dagger = 1$. The exterior transformations are not effected by this transformation. The constraint that the transformations are consistent with charge conjugation demands that Ψ and $\bar{\Psi}^\dagger$ transform in the same way, and this implies that S is an even element, comprising only terms with products of an even number of vectors. The accompanying similarity transformations $T_a' \to S^\dagger T_a' S$ and $P_{+3} \to S^\dagger P_{+3} S$ maintain the results, preserving the structure constants of the group algebras. It can also be shown that for any other set in which all the interior generators are written solely as bivectors, the same weak hypercharge assignments are obtained. In brief, if T_1' through T_8' of the new set are all bivectors, it can be shown that S must be generated by bivectors. Such a bivector transformation S on equation (36) maintains the same form. This framework then gives a geometric basis for the gauge group of the standard model, which arises unambiguously through the various rotational symmetries of the algebraic current in seven-dimensional space.

5. Higgs field

When looking at the exterior invariances of the current, we previously disregarded the higher-dimensional vector components and allowed them to freely rotate among each other. This Lorentz-invariant vector space is then a carrier space for the set of exterior gauge transformations and affords a natural inclusion of the minimal Higgs field [21]. With the help of the matrix representation (11), one can verify that by formulating the complex scalar isodoublet H and conjugate Higgs $H_\mathrm{c} = \bar{H}^\dagger$ as

$$\begin{aligned} H &= (-\phi_1e_6 + \phi_2e_7)P_{-\alpha} + (\phi_3e_5 - \phi_4e_4)P_{+\beta} \sim \left\lceil \begin{pmatrix} \phi_1 + \mathrm{i}\phi_2 \\ \phi_3 + \mathrm{i}\phi_4 \end{pmatrix} \right\rfloor \\ H_\mathrm{c} &= (\phi_1e_6 - \phi_2e_7)P_{+\alpha} - (\phi_3e_5 - \phi_4e_4)P_{-\beta} \sim \left\lceil \begin{pmatrix} \phi_3 - \mathrm{i}\phi_4 \\ -\phi_1 + \mathrm{i}\phi_2 \end{pmatrix} \right\rfloor \end{aligned} \tag{43}$$

where the ϕ_j are real scalars, the expression

$$\mathcal{L}_M = \frac{1}{\sqrt{2}}\langle \Psi^\dagger G_e H \Psi P_\ell + \Psi^\dagger (G_d H + G_u H_\mathrm{c})\Psi P_\mathrm{q}\rangle_\mathrm{S} \tag{44}$$

is identical to the conventional Higgs-coupling Lagrangian term with coupling strengths $G_{e,d,u}$. The projectors $P_\ell = P(1)$ and $P_\mathrm{q} = P(4) + P(5) + P(8)$ are used to separate the lepton and quark currents. The transformation required for gauge invariance,

$$H \to \exp(\theta_0 T_0 + \theta_a T_a) H \exp(-\theta_0 T_0 - \theta_b T_b) \tag{45}$$

is equivalent to the conventional notation

$$\begin{pmatrix} \phi^+ \\ \phi^0 \end{pmatrix} \to \exp(-\mathrm{i}Y\theta_0 - \mathrm{i}\theta_a\sigma_a/2)\begin{pmatrix} \phi^+ \\ \phi^0 \end{pmatrix} \tag{46}$$

where $\phi^+ \equiv \phi_1 + i\phi_2$ and $\phi^0 = \phi_3 + i\phi_4$. The weak hypercharge assignment of $Y = 1$ ($Y = -1$) for the Higgs field (conjugate field) has been recovered naturally from the double-sided algebraic transformation.

Note that H and H_c consist only of odd elements in the span of $\{e_4, e_5, e_6, e_7\}$ and therefore change the chirality, for example $P_R H = H P_L$. In fact, they exhaust the couplings between R and L leptons and between the R and L quarks. Application of the gauge transformation (45) using only the exterior generators of $SU(2)_L$ and $U(1)$ naturally separates the higher-dimensional vector space into the two invariant carrier spaces of H and H_c and ensures that the gauge fixing occurs consistently, reducing both H and H_c to a component along the same higher dimension. (We have essentially pre-aligned the Higgs with its conjugate field by using the same components for each.) On the other hand, the two generators of rotations that we excluded from possible gauge groups invariably mix the spaces of H and H_c. The symmetry of the Higgs coupling therefore requires the chiral asymmetry seen in the electroweak interaction. Ignoring the various weighted projectors used in both the Higgs field (43) and Lagrangian term (44) to give distinct masses to the different fermions, the form of equation (44) is the same as that of the current (14), but where the components of the current being extracted are from the set $\{e_4, e_5, e_6, e_7\}$. The Higgs field—one of the least understood aspects of the standard model—thus arises here simply as a coupling to the higher-dimensional vector components of the current.

6. Conclusion

We began by formulating a generalized current expression in the Clifford algebra $C\ell_7$ of seven-dimensional space. The addition of four spacelike dimensions to those of physical space is the minimum necessary to incorporate all the fermions of one generation, both particles and antiparticles, into a single spinorial element. By examining all possible rotations of the generalized current that leave the right-chiral neutrino (and left-chiral antineutrino) sterile and conserve the spacetime components of the particle and antiparticle currents, we found that they are precisely those of the gauge group of the standard model. The standard-model gauge symmetries are thus seen to be local rotation groups in the tangent space of a manifold with *only four* extra spacelike dimensions. In particular, the $SU(2)_L$ symmetries arise as *exterior* (left-sided) transformations representing rotations within the extra dimensions that act only on left-chiral fermions and their antiparticles, whereas the $SU(3)_C$ are *interior* (right-sided) ones that mix the colour charges of the quarks. The $U(1)_Y$ symmetry, complete with all the correct weak hypercharge assignments, arises as a unique group of double-sided rotations. All of these symmetries commute with the Poincaré group and are generated by bivectors of $C\ell_7$. The maximal chiral asymmetry of $SU(2)_L$ is required by the symmetry of the Higgs field.

We have further shown that in our model, rotations are in fact the only continuous transformations allowed that act entirely from the right or from the left. While we explained our model with the aid of a specific 8×8 matrix representation, the symmetries and weak hypercharge assignments depend only on the algebra and not on any particular selection of primitive idempotents or ordering of particle spinors in Ψ. Finally, the four extra dimensions required to fit a generation of fermions into a single algebraic spinor, together with their exterior transformation properties, are precisely what is needed for the four components of a minimal scalar Higgs field, again with the correct weak hypercharge assignment.

Many features of the standard model thus flow from the relatively simple geometry of seven-dimensional Euclidean space, but there are many other features that call for explanation, such as the origin of the three generations and the mass spectra. Work is continuing on these and other aspects of the standard model within the framework of our model.

Acknowledgments

One of us (GT) would like to thank H Georgi for suggestions on how to make the Clifford-algebra approach more accessible. This work was supported by the Natural Sciences and Engineering Research Council of Canada.

Appendix

Relations for the matrix representation adopted in this paper are summarized here. There are several versions of the Weyl representation. In this paper we have used

$$\gamma_0 = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \qquad \gamma_k = -\gamma^k = \begin{pmatrix} 0 & -\sigma_k \\ \sigma_k & 0 \end{pmatrix} \qquad \gamma_5 \equiv -\mathrm{i}\gamma_0\gamma_1\gamma_2\gamma_3 \tag{A.1}$$

with

$$\psi^{\mathrm{Weyl}} = \left[\begin{pmatrix} \psi_{\mathrm{R}} \\ \psi_{\mathrm{L}} \end{pmatrix} \right]. \tag{A.2}$$

This is consistent with [8].

The primitive projectors $P(n)$, which are represented by matrices with elements $P(n)_{jk} = \delta_{jn}\delta_{kn}$, are given as products of simple commuting projectors (15) by

$$\begin{aligned}
P(1) &= P_{+3}P_{+\alpha}P_{-\beta} = \bar{P}(6) \\
P(2) &= P_{-3}P_{+\alpha}P_{-\beta} = \bar{P}(5) \\
P(3) &= P_{-3}P_{-\alpha}P_{-\beta} = \bar{P}(8) \\
P(4) &= P_{+3}P_{-\alpha}P_{-\beta} = \bar{P}(7) \\
P(5) &= P_{+3}P_{-\alpha}P_{+\beta} = \bar{P}(2) \\
P(6) &= P_{-3}P_{-\alpha}P_{+\beta} = \bar{P}(1) \\
P(7) &= P_{-3}P_{+\alpha}P_{+\beta} = \bar{P}(4) \\
P(8) &= P_{+3}P_{+\alpha}P_{+\beta} = \bar{P}(3)\,.
\end{aligned} \tag{A.3}$$

Inverse relations, such as

$$P_{+3} = P(1) + P(4) + P(5) + P(8) \tag{A.4}$$

are easily obtained by summing and applying the complementarity of simple operators of opposite signs: $P_+ + P_- = 1$.

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